

Friday, October 17, Lecture 21: Residue Theorem

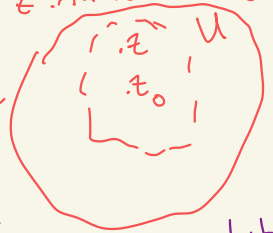
Last time...

- if a holomorphic f has a zero z_0 , we can factor it out:

$$f(z) = (z - z_0)^m \underset{\substack{\text{order of the zero} \\ \uparrow}}{g(z)}$$

holomorphic and $g(z) \neq 0$ for z in a nbhd of z_0

- $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ has a removable singularity if it extends to a holomorphic $f: \Omega \rightarrow \mathbb{C}$.
- f has a pole at z_0 if $\frac{1}{f}$ has a zero at z_0 .



we proved this last time

Def:

Let f have a pole of order m at z_0 , so we may write

$$f(z) = \underbrace{\frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{a_{-1}}{(z-z_0)}}_{\text{principal part of } f} + \underbrace{G(z)}_{\substack{\uparrow \\ \text{holomorphic} \\ \text{near } z_0}}$$

The residue of f at z_0 is

$$\text{Res}_{z_0} f := a_{-1}$$

Exs:

$$\bullet \text{Res}_0 \frac{1}{z} = 1.$$

$$\bullet \text{Res}_0 \frac{1}{z^2} = 0$$

Partial Fractions

• $\text{Res}_1 \frac{1}{z^2-1}$. Well,

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$$

hold at 1,
doesn't contribute
to $\text{Res}_1(G(z))$

This one does,

To compute: $\text{Res}_1 \frac{1}{z^2-1} = B = \lim_{z \rightarrow 1} (z-1) \frac{1}{(z^2-1)} = \lim_{z \rightarrow 1} \frac{1}{z+1} = \boxed{\frac{1}{2}}$

• $\text{Res}_\pi \frac{e^z}{\sin z}$. Well... $\sin z = \text{---}$ near $z=\pi$, $\sin z \approx -(z-\pi)$

Indeed, $z=\pi$ is a simple zero of $\sin z$ since
 $\frac{d}{dz}(\sin z)|_{z=\pi} = -1$.

$$\Rightarrow \frac{1}{\sin z} = \frac{\tilde{a}_{-1}}{(z-\pi)} + \tilde{g}(z). \text{ Since } e^z \text{ is holomorphic,}$$

this is the only pole, so

$$\frac{e^z}{\sin z} = \frac{a_{-1}}{z-\pi} + g(z), \text{ where } a_{-1} = \text{Res}_\pi \frac{e^z}{\sin z}$$

$$\text{Res}_\pi \frac{e^z}{\sin z} = \lim_{z \rightarrow \pi} (z-\pi) \cdot \frac{e^z}{\sin z} = \lim_{z \rightarrow \pi} \frac{(z-\pi)}{\sin z} \lim_{z \rightarrow \pi} e^z = -1 \cdot e^\pi = -e^\pi$$

Moral: if z_0 is a simple pole of f ,

$$\text{Res}_{z_0} f = \lim_{z \rightarrow z_0} (z-z_0)f.$$

(Messier for higher order poles, but same idea)

Does that form look familiar? HW Primto Residues

Thm (Residue Theorem, simple version)

Let f be holomorphic on $\Omega \supset \overline{D_R(z_0)}$, γ bounding circle, except for at z_0 . Then

$$\int_{\gamma} f(z) dz = 2\pi i \operatorname{Res}_{z_0}(f).$$

Pf: when z_0 is a pole^{*} of order m ,

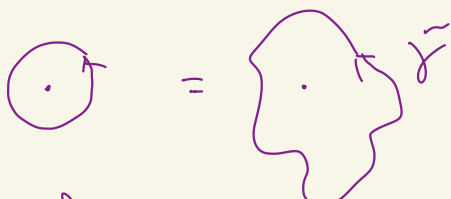
$$\begin{aligned} \int_{\gamma} f dz &= a_{-m} \int_{\gamma} \frac{1}{(z-z_0)^m} dz + a_{-m+1} \int_{\gamma} \frac{1}{(z-z_0)^{m-1}} dz + \dots + a_{-1} \int_{\gamma} \frac{1}{(z-z_0)} dz + \underbrace{\int_{\gamma} G(z) dz}_{=0 \text{ by Cauchy's thm}} \\ &\quad \underbrace{= 0, \text{ since } \int_{\text{unit circle}} \frac{1}{z^m} = 0 \text{ if } m \geq 1.} \end{aligned}$$

$$\begin{aligned} &= a_{-1} \underbrace{2\pi i}_{= \int_{\text{unit circle}} \frac{1}{z} dz} \\ &= (\operatorname{Res}_{z_0} f) 2\pi i. \end{aligned}$$

* We will encounter other types of singularities than poles. But the residue theorem still holds.



But by Cauchy's thm, we can deform:

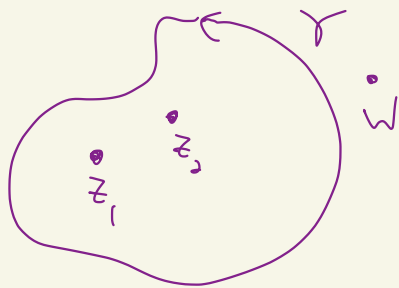

$$\int_{\gamma} f = \int_{\tilde{\gamma}} f$$

Thm (Residue Thm for simple closed contours)

Let Ω simply connected, containing simple loop γ (positively oriented)

Let $f: \Omega \rightarrow \mathbb{C}$ holomorphic except at a finite set of points $z_1, \dots, z_N$ inside γ . Then

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^N \text{Res}_{z_k} f.$$



$$\int_{\gamma} f(z) dz = 2\pi i (\text{Res}_{z_1} f + \text{Res}_{z_2} f)$$

(does not see w).

- This is an amazing theorem which we will spend Monday and Wednesday applying. But not yet.

A useful trick for computing residues:

Ex:

Find the residue of

$$f(z) = \frac{z \sin(z)}{(z-\pi)^3} = \frac{g(z)}{h(z)} \text{ at } z_0 = \pi,$$

well, we know since denominator has a zero of order 3,

$$f(z) = \frac{a_{-3}}{(z-\pi)^3} + \frac{a_{-2}}{(z-\pi)^2} + \frac{a_{-1}}{(z-\pi)} + a_0 + a_1 z + \dots$$

(It may be that some a_{-n} are 0). One way to get a_{-1} is by

$$(z-\pi)^3 f(z) = a_{-3} + a_{-2}(z-\pi) + a_{-1}(z-\pi)^2 + a_0(z-\pi)^3 + a_1(z-\pi)^4 + \dots$$

$$\text{So } \frac{d^2}{dz^2} \left[(z-\pi)^3 f(z) \right] \Big|_{z=\pi} = 2! a_{-1}$$

Compute:

$$\frac{d^2}{dz^2} \left((z-\pi)^3 \cdot \frac{z \sin z}{(z-\pi)^3} \right) \Big|_{z=\pi} = \frac{d^2}{dz^2} (z \sin z) \Big|_{z=\pi} = 2 \cos \pi - \pi \sin \pi = -2$$

$$\text{So } \operatorname{Res}_{\pi} f = a_{-1} = -1.$$

In general: if $f = \frac{g}{h}$ with g, h holomorphic, and

h has a zero z_0 of order m ,

$$\operatorname{Res}_{z_0} f = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m \frac{g}{h} \right) \Big|_{z=z_0}.$$

- So... we have removable singularities and poles.
what else can happen?

Ex: Consider $e^{1/z}$.

$$\begin{aligned} \exp\left(\frac{1}{z}\right) &= 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \dots + \frac{1}{n!}\left(\frac{1}{z}\right)^n \\ &= \dots + \frac{a_{-n}}{z^n} + \frac{a_{-n+1}}{z^{n-1}} + \dots + \frac{a_{-1}}{z} + 1 \end{aligned}$$

Notice this goes on forever in the $-n$ direction:

so $e^{1/z}$ does not have a pole at $z=0$, it

has something more dramatic: an essential singularity.

Def: Essential singularity

Let f be holomorphic on Ω except at a point z_0 .

We call z_0 an essential singularity if z_0 is neither a removable singularity nor a pole.

- The form of $e^{1/z}$ is sort of like a power series, but we allow for negative powers.
- In general these are crazy.